

THE RATE OF INCREASE FOR RECURSION WITH QUADRATIC NON-LINEARITY

E.Ostrovsky^a, L.Sirota^b

^a Corresponding Author. Department of Mathematics and computer science,
Bar-Ilan University, 84105, Ramat Gan, Israel.
E-mail: galo@list.ru eugostrovsky@list.ru

^b Department of Mathematics and computer science. Bar-Ilan University, 84105,
Ramat Gan, Israel.
E-mail: sirota3@bezeqint.net

Abstract.

We investigate in this short report the rate of increase of positive numerical recursion with quadratic non-linearity. More exactly, we intent to calculate the logarithmic index of its increasing.

We present also the possible application in the theory of the Navier-Stokes equations.

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1 Notations. Statement of problem. Conditions. Possible applications.

Let us consider the following numerical recurrence relation (dynamical system)

$$D(n+1) = a + b \cdot D^2(n), \quad a, b = \text{const} > 0, \quad n = 0, 1, 2, \dots, \quad D(n) = D_{a,b}(n). \quad (1.1)$$

We can and will suppose without loss of generality $D(0) = 1$ (initial condition). We impose in the sequel on the parameters a, b the following conditions:

$$b > 0; \quad a \cdot b \geq 1/4. \quad (1.2)$$

This conditions guarantee the monotonic increasing of the sequence $D(n)$:
 $D(n+1) \geq D(n)$.

Something similar occurred in the theory of elliptical curves and following in the coding theory [16]. Another applications and investigations of these equations are discussed in the book [15].

The authors are faced with this kind of equation by the investigation of numerical method for Navier-Stokes equation, see [13], [14]. We describe in greater detail.

The mild solution $u = u(x, t)$ of a Navier-Stokes equation in the whole space $x \in R^d$ throughout its lifetime $t \in [0, T]$, $0 < T = \text{const} \leq \infty$ may be represented as a limit as $n \rightarrow \infty$, $n = 0, 1, 2, \dots$ the following recursion:

$$u_{n+1}(x, t) = u_0(x, t) + G[u_n, u_n](x, t), n = 0, 1, 2, \dots,$$

where $u_0(x, t)$ is the solution of heat equation with correspondent initial value and right-hand side and $G[u, v]$ is bilinear unbounded pseudo-differential operator, [6], [7]. See also the articles [1], [2], [3], [4], [5], [8], [9], [10], [11] etc. The second iteration is investigated in [12].

Recall that the function $u = u(x, t)$ and hence the functions $u_n(x, t)$, $n = 0, 1, 2, \dots$ are vector functions:

$$u(x, t) = \vec{u}(x, t) = \{u^{(i)}(x, t)\}, i = 1, 2, \dots, d; \quad (1.3)$$

therefore the functional $G[u, v] = G[\vec{u}, \vec{v}] = \vec{G}[\vec{u}, \vec{v}]$ may be interpreted as a tensor:

$$\vec{G} = \{g_{i,j}^m\}, \quad \vec{G}[\vec{u}, \vec{v}]_m = \sum_{i=1}^d \sum_{j=1}^d g_{i,j}^m u^{(i)} v^{(j)}, \quad m = 1, 2, \dots, d. \quad (1.4)$$

We denote by $D(n)$ the amount of *independent* summands in the expression for the n^{th} iteration:

$$u_n^{(i)} = \sum_{s_1=1}^{D(n)} \omega_{n,s_1}^{(i)}, \quad v_n^{(j)} = \sum_{s_2=1}^{D(n)} \kappa_{n,s_2}^{(j)}, \quad \omega_{n,s_1}, \kappa_{n,s_2} = \omega_{n,s_1}(x, t), \kappa_{n,s_2}(x, t), \quad (1.5)$$

then

$$\vec{G}[\vec{u}_n, \vec{v}_n]_m = \sum_{i=1}^d \sum_{j=1}^d g_{i,j}^m \sum_{s_1=1}^{D(n)} \sum_{s_2=1}^{D(n)} \omega_{n,s_1}^{(i)} \kappa_{n,s_2}^{(j)} u_n^{(i)} v_n^{(j)}, \quad k = 1, 2, \dots, d. \quad (1.6)$$

The last expression contains exactly in general case $d^2 \cdot D^2(n)$ independent summands.

Obviously for all the values k $D(0) = 1$ and

$$D(n+1) = 1 + d^2 \cdot D^2(n), \quad (1.7)$$

i.e. in this case $a = 1$, $b = d$. Since $d \geq 1$, the conditions (1.2) are satisfied.

Our claim in this report is investigation of recurrence equation (1.1) under condition (1.2): obtaining of upper and lower bounds and calculating the asymptotic for the solution.

2 Main results: bilateral bounds and asymptotic behavior for solution.

Theorem.

$$\forall k, l = 1, 2, \dots \Rightarrow 1 \leq \frac{b D(k+l)}{[b D(l)]^{2^k}} \leq \left[1 + \frac{a}{b D^2(l)} \right]^{2^k - 1}; \quad (2.1)$$

$$\forall k \geq 1 \Rightarrow \lim_{l \rightarrow \infty} \frac{b D(k+l)}{[b D(l)]^{2^k}} = 1. \quad (2.2)$$

Proof. *Lower bound:*

$$D(k+1) \geq b D^2(k), \quad k = l, l+1, l+2, \dots; \quad l = \text{const} = 0, 1, \dots$$

We deduce:

$$D(l+1) \geq b D^2(l), \quad D(l+2) \geq b^3 D^4(l), \quad D(l+3) \geq b^7 D^8(l), \dots$$

By induction:

$$D(l+k) \geq b^{2^k - 1} D^{2^k}(l). \quad (2.3)$$

Upper bound. We deduce denoting

$$Q(l) = 1 + \frac{a}{b D^2(l)}$$

and taking into account the monotonicity of the sequence $D_{a,b}(n)$: if $k \geq l$ then

$$D(k+1) = a + b D^2(k) = b D^2(k) \left(1 + \frac{a}{b D^2(k)} \right) \leq$$

$$b D^2(k) \left(1 + \frac{a}{b D^2(l)} \right) = b D^2(k) Q(l),$$

and we find analogously

$$D(k+l) \leq b^{2^k - 1} D^{2^k}(l) [Q(l)]^{2^k - 1}. \quad (2.4)$$

The assertion (2.2) follows immediately from the bilateral estimates (2.1).

3 Examples.

We intent to illustrate by building of some numerical examples the huge growth rate $D(n)$ to infinity.

1. As regards to the Navier-Stokes equation in real case.

Here $d = 3$; i.e. $D(n) = D_{1,9}(n)$; $D(0) = 1$, $D(n+1) = 1 + 9D^2(n)$:

$$D(0) = 1, D(1) = 10, D(2) = 901, D(3) = 811\ 802, D(4) = 659\ 022\ 487\ 205,$$

$$D(5) = 434\ 310\ 638\ 641\ 864\ 388\ 712\ 026, D(6) \approx 1.886257308 \cdot 10^{47},$$

$$D(7) \approx 3.5579666 \cdot 10^{94}.$$

2. Let now $a = b = 1$; i.e. $D(n) = D_{1,1}(n)$; then

$$D(0) = 1, D(1) = 2, D(2) = 5, D(3) = 26, D(4) = 677, D(5) = 458\ 330,$$

$$D(6) = 210\ 066\ 388\ 901, D(7) = 44\ 127\ 887\ 745\ 906\ 175\ 987\ 802.$$

3. For comparison:

$$D(n) \geq \tilde{D}(n) := 2^{(2^n - 1)};$$

$$\tilde{D}(0) = 1, \tilde{D}(1) = 2, \tilde{D}(2) = 4, \tilde{D}(3) = 16, \tilde{D}(4) = 256, \tilde{D}(5) = 65\ 536,$$

$$\tilde{D}(6) = 4\ 294\ 967\ 296, \tilde{D}(7) = 18\ 446\ 744\ 073\ 709\ 551\ 616.$$

The great difference between $D_{1,1}(n)$ and $\tilde{D}(n)$ show us the influence of free member "a" in the source equation (1.1).

4 Concluding remarks.

A. At the same method may be used by investigation of the non-linear recursion

$$D(n+1) = F(n, D(n))$$

with monotonic increasing power of non-linearity such that

$$C_1 z^{1+\Delta} \leq F(n, z) \leq C_2 z^{1+\Delta}, \quad z \geq 1, \quad \Delta = \text{const} > 0.$$

B. The vector analog of the equation (1.1) has a form

$$D(n+1) = \vec{a} + D^2(n)\vec{b},$$

where $D(n)$ is the square matrix $m \times m$ and $\dim \vec{a} = \dim \vec{b} = m$, $m = 2, 3, \dots$

C. Obviously, if in addition both the numbers a and b are integer, then quite sequence $\{D(n)\}$ is integer. Therefore

$$\forall k, l = 1, 2, \dots \Rightarrow b^{2^k-1} D(l)^{2^k} \leq D(k+l) \leq$$

$$\text{Ent} \left\{ \left[1 + \frac{a}{bD^2(l)} \right]^{2^k-1} \cdot b^{2^k-1} \cdot D(l)^{2^k} \right\},$$

where $\text{Ent}(z)$ denotes the integer part of the real number z , since $D(n)$ is integer sequence.

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